| | |
|---|---|
| Title | Algebraic independence properties of the values of Hecke-Mahler series and its derivatives (Analytic Number Theory and Related Areas) |
| Author(s) | Tanuma, Yusuke |
| Citation | 数理解析研究所講究録 = RIMS Kokyuroku (2018), 2092: 75-87 |
| Issue Date | 2018-11 |
| URL | http://hdl.handle.net/2433/251652 |
| Right | |
| Type | Departmental Bulletin Paper |
| Textversion | publisher |

数理解析研究所講究録 第2092巻 2018年 75-87



# Algebraic independence properties of the values of Hecke-Mahler series and its derivatives

慶應義塾大学大学院 理工学研究科 基礎理工学専攻 田沼 優佑
Yusuke Tanuma
Graduate School of Science and Technology,
Keio University

## 1 Introduction

This article is based on [11], joint work with Professor Taka-aki Tanaka. Let $\omega$ be a real number. We denote by $[x]$ the integral part of the real number $x$, namely the largest integer not exceeding $x$. Hecke-Mahler series, the generating function of the sequence $\{[k\omega]\}_{k=1}^{\infty}$, is defined by

$$h_{\omega}(z)=\sum_{k=1}^{\infty}[k\omega]z^{k},$$

where $z$ is complex with $|z|<1$. Hecke [2] proved that, if $\omega$ is an irrational number, then $h_{\omega}(z)$ has the unit circle $|z|=1$ as its natural boundary. Mahler [5] proved that, if $\omega$ is a quadratic irrational number, then the value $h_{\omega}(\alpha)$ is transcendental, where $\alpha$ is a nonzero algebraic number inside the unit circle.

In what follows, let $\omega$ be a real quadratic irrational number. We denote by $h_{\omega}^{(l)}(z)$ the derivative of $h_{\omega}(z)$ of order $l$. Nishioka proved the algebraic independence of the values of $h_{\omega}(z)$ and its derivative of any order at any fixed nonzero algebraic number inside the unit circle.

**Theorem 1** (Nishioka [8]). *If $\alpha$ is an algebraic number with $0<|\alpha|<1$, then the infinite set of the values $\{h_{\omega}^{(l)}(\alpha)\mid l\geq 0\}$ is algebraically independent.*

On the other hand, Masser proved the algebraic independence of the values of $h_{\omega}(z)$ at any nonzero distinct algebraic numbers inside the unit circle.

**Theorem 2** (Masser [6]). *The infinite set of the values $\{h_{\omega}(\alpha)\mid\alpha\in\overline{\mathbb{Q}},\ 0<|\alpha|<1\}$ is algebraically independent.*

We denote by $\omega'$ the conjugate of the real quadratic irrational number $\omega$. The following is the main theorem of this article.

**Theorem 3** (with Tanaka [11]). *Suppose that $\omega$ satisfies $|\omega-\omega'|>2$. Then the infinite set of the values $\{h_\omega^{(l)}(\alpha)\mid l\ge 0,\ \alpha\in\overline{\mathbb{Q}},\ 0<|\alpha|<1\}$ is algebraically independent.*

**Corollary 1.** *Suppose that $\omega$ is an algebraic integer. Then the infinite set of the values $\{h_\omega^{(l)}(\alpha)\mid l\ge 0,\ \alpha\in\overline{\mathbb{Q}},\ 0<|\alpha|<1\}$ is algebraically independent.*

**Corollary 2.** *Let $m>1$ be a square-free integer and $r$ a rational number. Put $\omega=r\sqrt{m}$. If $|\omega|>1$, then the infinite set of the values $\{h_\omega^{(l)}(\alpha)\mid l\ge 0,\ \alpha\in\overline{\mathbb{Q}},\ 0<|\alpha|<1\}$ is algebraically independent.*

For more general irrational number $\omega$, some results on the arithmetic properties of the values of $h_\omega(z)$ can be found in for example [3], [7], [1].

In the next section we consider the case where $\omega$ is rational. In Section 3 we give the sketch of the proof of Theorem 3.

## 2 On the case where $\omega$ is rational

For any positive number $\omega$, we define

$$H_\omega(z_1,z_2)=\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{[k_1\omega]}z_1^{k_1}z_2^{k_2}.$$

As mentioned in the previous section, if $\omega$ is an irrational number, then $h_\omega(z)$ is transcendental over $\mathbb{C}(z)$. In the rest of this section, let $\omega$ be a rational number, not necessarily positive. We assume that $\omega$ is expanded in the finite continued fraction

$$\omega=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\cfrac{1}{a_r}}}}=:[a_0;a_1,a_2,\dots,a_r],$$

where $a_0=[\omega]$ and $a_1,\dots,a_r$ are positive integers. We denote by $\{x\}$ the fractional part of the real number $x$. Put $\chi=\{\omega\}=[0;a_1,a_2,\dots,a_r]$. Define positive integers $s_\mu,t_\mu$ $(0\le\mu\le r-1)$ by $\chi=s_0/t_0$,

$$\frac{s_\mu}{t_\mu}=\frac{1}{a_{\mu+1}+s_{\mu+1}/t_{\mu+1}}\qquad(0\le\mu\le r-2)$$

and $s_{r-1}/t_{r-1}=1/a_r$ with $s_\mu$ and $t_\mu$ relatively prime for any $\mu$. Define positive integers $p_\mu$, $q_\mu$ $(0\le\mu\le r)$ by

$$\begin{pmatrix}p_\mu & q_\mu\\ p_{\mu-1} & q_{\mu-1}\end{pmatrix}=\begin{pmatrix}a_\mu & 1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}a_1 & 1\\ 1 & 0\end{pmatrix}.$$

For any positive integer $a$, we have

$$\begin{aligned}
H_{a+\omega}(z_1, z_2) &= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{[k_1 a + k_1\omega]} z_1^{k_1} z_2^{k_2} \\
&= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{k_1 a + [k_1\omega]} z_1^{k_1} z_2^{k_2} \\
&= \sum_{k_1=1}^{\infty} \left( z_1^{k_1} \sum_{k_2=1}^{ak_1} z_2^{k_2} + (z_1 z_2^a)^{k_1} \sum_{k_2=1}^{[k_1\omega]} z_2^{k_2} \right) \\
&= \sum_{k_1=1}^{\infty} z_1^{k_1} \frac{z_2 - z_2^{ak_1+1}}{1 - z_2} + H_\omega(z_1 z_2^a, z_2) \\
&= \frac{z_1 z_2}{(1-z_1)(1-z_2)} - \frac{z_1 z_2^{a+1}}{(1-z_2)(1-z_1 z_2^a)} + H_\omega(z_1 z_2^a, z_2). \qquad (1)
\end{aligned}$$

For any rational number $p/q$, where $p, q$ are relatively prime positive integers,

$$\begin{aligned}
H_{p/q}(z_1, z_2) + H_{q/p}(z_2, z_1) &= \sum_{\substack{k_1 \geq 1, k_2 \geq 1 \\ k_2 \leq k_1 p/q}} z_1^{k_1} z_2^{k_2} + \sum_{\substack{k_1 \geq 1, k_2 \geq 1 \\ k_2 \leq k_1 q/p}} z_2^{k_1} z_1^{k_2} \\
&= \sum_{\substack{k_1 \geq 1, k_2 \geq 1 \\ k_2 \leq k_1 p/q}} z_1^{k_1} z_2^{k_2} + \sum_{\substack{k_1 \geq 1, k_2 \geq 1 \\ k_1 \geq k_2 p/q}} z_1^{k_2} z_2^{k_1} \\
&= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} z_1^{k_1} z_2^{k_2} + \sum_{k=1}^{\infty} (z_1^q z_2^p)^k \\
&= \frac{z_1 z_2}{(1-z_1)(1-z_2)} + \frac{z_1^q z_2^p}{1 - z_1^q z_2^p}. \qquad (2)
\end{aligned}$$

Hence

$$\begin{aligned}
H_{s_\mu/t_\mu}(z_1, z_2) &= \frac{z_1 z_2}{(1-z_1)(1-z_2)} + \frac{z_1^{t_\mu} z_2^{s_\mu}}{1 - z_1^{t_\mu} z_2^{s_\mu}} - H_{a_{\mu+1} + s_{\mu+1}/t_{\mu+1}}(z_2, z_1) \\
&= \frac{z_1^{a_{\mu+1}+1} z_2}{(1-z_1)(1 - z_1^{a_{\mu+1}} z_2)} + \frac{z_1^{t_\mu} z_2^{s_\mu}}{1 - z_1^{t_\mu} z_2^{s_\mu}} - H_{s_{\mu+1}/t_{\mu+1}}(z_1^{a_{\mu+1}} z_2, z_1).
\end{aligned}$$

Therefore, noting the definition of $p_\mu$ and $q_\mu$ $(0 \le \mu \le 2)$, we see that

$$
\begin{aligned}
&H_\chi(z_1,z_2) = H_{s_0/t_0}(z_1,z_2)\\
= &\frac{z_1^{a_1+1}z_2}{(1-z_1)(1-z_1^{a_1}z_2)} + \frac{z_1^{t_0}z_2^{s_0}}{1-z_1^{t_0}z_2^{s_0}} - H_{s_1/t_1}(z_1^{a_1}z_2, z_1)\\
= &\frac{z_1^{p_1+p_0}z_2^{q_1}}{(1-z_1^{p_0})(1-z_1^{p_1}z_2^{q_1})} + \frac{z_1^{t_0}z_2^{s_0}}{1-z_1^{t_0}z_2^{s_0}} - H_{s_1/t_1}(z_1^{p_1}z_2^{q_1}, z_1^{p_0})\\
= &\frac{z_1^{p_1+p_0}z_2^{q_1}}{(1-z_1^{p_0})(1-z_1^{p_1}z_2^{q_1})} + \frac{z_1^{t_0}z_2^{s_0}}{1-z_1^{t_0}z_2^{s_0}} - \frac{(z_1^{p_1}z_2^{q_1})^{a_2+1}z_1^{p_0}}{(1-z_1^{p_1}z_2^{q_1})(1-(z_1^{p_1}z_2^{q_1})^{a_2}z_1^{p_0})}\\
&-\frac{(z_1^{p_1}z_2^{q_1})^{t_1}z_1^{p_0s_1}}{1-(z_1^{p_1}z_2^{q_1})^{t_1}z_1^{p_0s_1}} + H_{s_2/t_2}((z_1^{p_1}z_2^{q_1})^{a_2}z_1^{p_0}, z_1^{p_1}z_2^{q_1})\\
= &\frac{z_1^{p_1+p_0}z_2^{q_1}}{(1-z_1^{p_0})(1-z_1^{p_1}z_2^{q_1})} + \frac{z_1^{t_0}z_2^{s_0}}{1-z_1^{t_0}z_2^{s_0}} - \frac{z_1^{p_2+p_1}z_2^{q_2+q_1}}{(1-z_1^{p_1}z_2^{q_1})(1-z_1^{p_2}z_2^{q_2})}\\
&-\frac{z_1^{p_1t_1+p_0s_1}z_2^{q_1t_1}}{1-z_1^{p_1t_1+p_0s_1}z_2^{q_1t_1}} + H_{s_2/t_2}(z_1^{p_2}z_2^{q_2}, z_1^{p_1}z_2^{q_1}).
\end{aligned}
$$

Continuing this process, we see that

$$
\begin{aligned}
&H_\chi(z_1,z_2)\\
= &\sum_{\mu=0}^{r-2}(-1)^\mu\left(\frac{z_1^{p_{\mu+1}+p_\mu}z_2^{q_{\mu+1}+q_\mu}}{(1-z_1^{p_{\mu+1}}z_2^{q_{\mu+1}})(1-z_1^{p_\mu}z_2^{q_\mu})} + \frac{z_1^{p_\mu t_\mu+p_{\mu-1}s_\mu}z_2^{q_\mu t_\mu+q_{\mu-1}s_\mu}}{1-z_1^{p_\mu t_\mu+p_{\mu-1}s_\mu}z_2^{q_\mu t_\mu+q_{\mu-1}s_\mu}}\right)\\
&+(-1)^{r-1}H_{s_{r-1}/t_{r-1}}(z_1^{p_{r-1}}z_2^{q_{r-1}}, z_1^{p_{r-2}}z_2^{q_{r-2}}).
\end{aligned}
$$

Since

$$
\begin{aligned}
H_{s_{r-1}/t_{r-1}}(z_1,z_2) &= \frac{z_1z_2}{(1-z_1)(1-z_2)} + \frac{z_1^{t_{r-1}}z_2^{s_{r-1}}}{1-z_1^{t_{r-1}}z_2^{s_{r-1}}} - H_{a_r}(z_2,z_1)\\
&= \frac{z_1^{a_r+1}z_2}{(1-z_1)(1-z_1^{a_r}z_2)} + \frac{z_1^{t_{r-1}}z_2^{s_{r-1}}}{1-z_1^{t_{r-1}}z_2^{s_{r-1}}}
\end{aligned}
$$

by (2), $s_{r-1}/t_{r-1} = 1/a_r$ and (1) with $\omega = 0$, we have

$$
\begin{aligned}
&H_\chi(z_1,z_2)\\
= &\sum_{\mu=0}^{r-1}(-1)^\mu\left(\frac{z_1^{p_{\mu+1}+p_\mu}z_2^{q_{\mu+1}+q_\mu}}{(1-z_1^{p_{\mu+1}}z_2^{q_{\mu+1}})(1-z_1^{p_\mu}z_2^{q_\mu})} + \frac{z_1^{p_\mu t_\mu+p_{\mu-1}s_\mu}z_2^{q_\mu t_\mu+q_{\mu-1}s_\mu}}{1-z_1^{p_\mu t_\mu+p_{\mu-1}s_\mu}z_2^{q_\mu t_\mu+q_{\mu-1}s_\mu}}\right).
\end{aligned}
$$

Noting that

$$H_\chi(z,1) = h_\chi(z) = h_\omega(z) - \frac{a_0z}{(1-z)^2}$$

by $\chi = \omega - a_0$, we see that

$$h_\omega(z) = \frac{a_0z}{(1-z)^2} + \sum_{\mu=0}^{r-1}(-1)^\mu\left(\frac{z^{p_{\mu+1}+p_\mu}}{(1-z^{p_{\mu+1}})(1-z^{p_\mu})} + \frac{z^{p_\mu t_\mu+p_{\mu-1}s_\mu}}{1-z^{p_\mu t_\mu+p_{\mu-1}s_\mu}}\right) \in \mathbb{Q}(z).$$

Hence we see that $h_\omega(z)$ is a rational function if $\omega$ is rational.

# 3 Proof of Theorem 3

Let $\Omega = (\omega_{ij})$ be an $n\times n$ matrix with nonnegative integer entries. For $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, we define a multiplicative transformation $\Omega : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Omega\boldsymbol{z} = \left(\prod_{j=1}^n z_j^{\omega_{1j}}, \prod_{j=1}^n z_j^{\omega_{2j}}, \dots, \prod_{j=1}^n z_j^{\omega_{nj}}\right). \tag{3}$$

Then the iterates $\Omega^k\boldsymbol{z}$ $(k = 0, 1, 2, \dots)$ are well-defined.

For any positive irrational number $\chi$, we see that

$$\begin{aligned} H_\chi(z_1, z_2) + H_{1/\chi}(z_2, z_1) &= \sum_{\substack{h_1\geq 1, h_2\geq 1\\ h_2<h_1\chi}} z_1^{h_1} z_2^{h_2} + \sum_{\substack{h_1\geq 1, h_2\geq 1\\ h_1>h_2\chi}} z_1^{h_2} z_2^{h_1} \\ &= \frac{z_1}{1-z_1}\frac{z_2}{1-z_2}. \end{aligned} \tag{4}$$

Let $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $E(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ for any positive integer $a$. Define $\begin{pmatrix} p & q \\ r & s \end{pmatrix}\chi = (p\chi + q)/(r\chi + s)$, where $p, q, r, s$ are nonnegative integers. Then we see that

$$H_{D\chi}(z_1, z_2) \equiv -H_\chi(D(z_1, z_2)) \pmod{\mathbb{Q}(z_1, z_2)} \tag{5}$$

and that

$$H_{E(a)\chi}(z_1, z_2) \equiv H_\chi(E(a)(z_1, z_2)) \pmod{\mathbb{Q}(z_1, z_2)} \tag{6}$$

by (4) and (1), respectively, where $D(z_1, z_2)$ and $E(a)(z_1, z_2)$ are defined by (3).

***Sketch of the proof of Theorem 3.*** Since

$$\begin{aligned} h_\omega(z) + h_{-\omega}(z) &= \sum_{k=1}^\infty [k\omega]z^k + \sum_{k=1}^\infty [-k\omega]z^k \\ &= \sum_{k=1}^\infty [k\omega]z^k + \sum_{k=1}^\infty (-[k\omega] - 1)\, z^k = -\frac{z}{1-z}, \end{aligned}$$

we see that the algebraic independency of $\{h_\omega^{(l)}(\alpha) \mid l \geq 0,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is equivalent to that of $\{h_{-\omega}^{(l)}(\alpha) \mid l \geq 0,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$. Hence, considering $-\omega$ instead of $\omega$ if necessary, we may assume that $\omega > \omega'$. Since $|\omega - \omega'| > 2$, there exists an integer $a_0$ such that $0 < \omega - a_0 < 1$ and $\omega' - a_0 < -1$. Let $\chi = \omega - a_0$. By

$$h_\chi(z) = \sum_{k=1}^\infty [k(\omega - a_0)]z^k = h_\omega(z) - \frac{a_0 z}{(1-z)^2},$$

we may consider $\chi$ instead of $\omega$ by the same reason as above. Then $\chi$ is reduced and so expanded in a purely periodic continued fraction as follows:

$$\chi = [0; a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}},$$

where $a_1, a_2, \ldots$ are positive integers. Let $\nu$ be its even period. Then

$$\chi = [0; a_1, a_2, \ldots, a_\nu, \chi] = DE(a_1)DE(a_2)\cdots DE(a_\nu)\chi.$$

Let $T^{(1)} = E(a_\nu)DE(a_{\nu-1})D\cdots E(a_1)D$. Then by (5) and (6) we have

$$H_\chi(z_1, z_2) \equiv H_\chi(T^{(1)}(z_1, z_2)) \pmod{\mathbb{Q}(z_1, z_2)}.$$

Let $\alpha_1, \ldots, \alpha_n$ be any nonzero distinct algebraic numbers with $|\alpha_1|, \ldots, |\alpha_n| < 1$. It is enough to show that $\{h_\chi^{(l)}(\alpha_i) \mid 0 \le l \le L,\ 1 \le i \le n\}$ is algebraically independent for any sufficiently large $L$. Let $g_l(z) = \sum_{k=1}^\infty k^l [k\chi] z^k$. Then the algebraic independency of $\{h_\chi^{(l)}(\alpha_i) \mid 0 \le l \le L,\ 1 \le i \le n\}$ is equivalent to that of $\{g_l(\alpha_i) \mid 0 \le l \le L,\ 1 \le i \le n\}$. For the $\alpha_1, \ldots, \alpha_n$, there exist multiplicatively independent algebraic numbers $\beta_1, \ldots, \beta_m$ with $0 < |\beta_j| < 1$ $(1 \le j \le m)$ such that

$$\alpha_i = \zeta_i \prod_{j=1}^m \beta_j^{\ell_{ij}} \qquad (1 \le i \le n),$$

where $\zeta_i$ $(1 \le i \le n)$ are roots of unity and $\ell_{ij}$ $(1 \le i \le n,\ 1 \le j \le m)$ are nonnegative integers (cf. [4, Lemma 3]). We define

$$T^{(m)} = \mathrm{diag}\,(\underbrace{T^{(1)}, \ldots, T^{(1)}}_{m}).$$

Let $\boldsymbol{x} = (x_1, \ldots, x_m)$, $\boldsymbol{y} = (y_1, \ldots, y_m)$ be variables. Let

$$\boldsymbol{z}_0 = (\beta_1, 1, \beta_2, 1, \ldots, \beta_m, 1). \tag{7}$$

and $M_i(\boldsymbol{x}) = x_1^{\ell_{i1}} \cdots x_m^{\ell_{im}}$. Define

$$\begin{aligned} G_i(\boldsymbol{z}) = G(\zeta_i, M_i, \boldsymbol{z}) &:= H_\chi(\zeta_i M_i(\boldsymbol{x}), M_i(\boldsymbol{y})) \\ &= \sum_{k_1=1}^\infty \sum_{k_2=1}^{[k_1\chi]} (\zeta_i M_i(\boldsymbol{x}))^{k_1} M_i(\boldsymbol{y})^{k_2} \quad (1 \le i \le n), \end{aligned} \tag{8}$$

where $\boldsymbol{z} = (x_1, y_1, x_2, y_2, \ldots, x_m, y_m)$. By (8) we see that

$$D_{j_i}^l G_i(\boldsymbol{z}_0) = \sum_{h=1}^\infty \ell_{ij_i}^l h^l [h\chi] \alpha_i^h,$$

where $\ell_{ij_i} > 0$. Hence the algebraic independency of $\{g_l(\alpha_i) \mid 0 \le l \le L,\ 1 \le i \le n\}$ is equivalent to that of $\{D_{j_i}^l G_i(\boldsymbol{z}_0) \mid 0 \le l \le L,\ 1 \le i \le n\}$.

Similarly to $H_\chi$, each $G_i$ satisfies a functional equation:

**Lemma 1** (Masser [6, Lemma 3.3]). *There exists a positive power $T$ of $T^{(m)}$ such that*

$$G_i(\boldsymbol{z}) \equiv G_i(T\boldsymbol{z}) \qquad (\mathrm{mod}\ \overline{\mathbb{Q}}(\boldsymbol{z}))$$

*for any $i$ $(1 \le i \le n)$.*

The matrix $T$ in Lemma 1 can be written as

$$T = \mathrm{diag}\Bigg(\underbrace{\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}, \dots, \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}}_{m}\Bigg).$$

Let $D_j = x_j\partial/\partial x_j$ and $D'_j = y_j\partial/\partial y_j$ $(1 \le j \le m)$. Since

$$\begin{aligned} D_jG_i(\boldsymbol{z}) &\equiv x_j\frac{\partial G_i}{\partial x_j}(T\boldsymbol{z})t_{11}x_j^{t_{11}-1}y_j^{t_{12}} + x_j\frac{\partial G_i}{\partial y_j}(T\boldsymbol{z})t_{21}x_j^{t_{21}-1}y_j^{t_{22}} && (\mathrm{mod}\ \overline{\mathbb{Q}}(\boldsymbol{z})) \\ &\equiv t_{11}D_jG_i(T\boldsymbol{z}) + t_{21}D'_jG_i(T\boldsymbol{z}) && (\mathrm{mod}\ \overline{\mathbb{Q}}(\boldsymbol{z})) \end{aligned}$$

and

$$\begin{aligned} D'_jG_i(\boldsymbol{z}) &\equiv y_j\frac{\partial G_i}{\partial x_j}(T\boldsymbol{z})t_{12}x_j^{t_{11}}y_j^{t_{12}-1} + y_j\frac{\partial G_i}{\partial y_j}(T\boldsymbol{z})t_{22}x_j^{t_{21}}y_j^{t_{22}-1} && (\mathrm{mod}\ \overline{\mathbb{Q}}(\boldsymbol{z})) \\ &\equiv t_{12}D_jG_i(T\boldsymbol{z}) + t_{22}D'_jG_i(T\boldsymbol{z}) && (\mathrm{mod}\ \overline{\mathbb{Q}}(\boldsymbol{z})) \end{aligned}$$

for $1 \le i \le n$, $1 \le j \le m$, we see that $D_1^{k_1}{D'_1}^{k'_1}\cdots D_m^{k_m}{D'_m}^{k'_m}G_i(\boldsymbol{z})$ $(0 \le k_1, k'_1, \dots, k_m, k'_m \le L,\ 1 \le i \le n)$ satisfy a system of functional equations of the form

$$\begin{aligned} &\begin{pmatrix} G_i(\boldsymbol{z}) \\ D_1G_i(\boldsymbol{z}) \\ \vdots \\ (D_1D'_1\cdots D_mD'_m)^LG_i(\boldsymbol{z}) \end{pmatrix} \\ \equiv\ A &\begin{pmatrix} G_i(T\boldsymbol{z}) \\ D_1G_i(T\boldsymbol{z}) \\ \vdots \\ (D_1D'_1\cdots D_mD'_m)^LG_i(T\boldsymbol{z}) \end{pmatrix} \quad \left(\mathrm{mod}\ \left(\overline{\mathbb{Q}}(\boldsymbol{z})\right)^{(L+1)^{2m}}\right) \end{aligned}$$

for $1 \le i \le n$ and for any $L \ge 0$, where $A$ is an $(L+1)^{2m} \times (L+1)^{2m}$ matrix with rational entries. In order to prove the algebraic independency of $\{D_{j_i}^{l}G_i(\boldsymbol{z}_0) \mid 0 \le l \le L,\ 1 \le i \le n\}$, we use the following criterion:

**Lemma 2** (Nishioka [9]). *Let $K$ be an algebraic number field. Suppose that $f_1(\boldsymbol{z}), \dots, f_M(\boldsymbol{z}) \in K[[z_1, \dots, z_N]]$ converge in an $N$-polydisc $U$ around the origin of $\mathbb{C}^N$ and satisfy the system of functional equations of the form*

$$\begin{pmatrix} f_1(\boldsymbol{z}) \\ \vdots \\ f_M(\boldsymbol{z}) \end{pmatrix} = A\begin{pmatrix} f_1(\Omega\boldsymbol{z}) \\ \vdots \\ f_M(\Omega\boldsymbol{z}) \end{pmatrix} + \begin{pmatrix} b_1(\boldsymbol{z}) \\ \vdots \\ b_M(\boldsymbol{z}) \end{pmatrix},$$

*where $A$ is an $M \times M$ matrix with entries in $K$ and $b_i(\boldsymbol{z}) \in K(z_1, \dots, z_N)$ $(1 \le i \le M)$. Let $\boldsymbol{\alpha}$ be a point in $U$ whose components are nonzero algebraic numbers. Assume that $\Omega$ and $\boldsymbol{\alpha}$ satisfy suitable conditions. Then, if $f_1(\boldsymbol{z}), \dots, f_r(\boldsymbol{z})$ $(r \le M)$ are linearly independent over $K$ modulo $K(z_1, \dots, z_N)$, then $f_1(\boldsymbol{\alpha}), \dots, f_r(\boldsymbol{\alpha})$ are algebraically independent.*

We can find that the matrix $T$ and the point $\boldsymbol{z}_0$ satisfy the condition in above lemma. Therefore it suffices to show that $\{D_{j_i}^l G_i(\boldsymbol{z}) \mid 0 \le l \le L,\ 1 \le i \le n\}$ is linearly independent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}(\boldsymbol{z})$.

On the contrary, we assume that $\{D_{j_i}^l G_i(\boldsymbol{z}) \mid 0 \le l \le L,\ 1 \le i \le n\}$ is linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}(\boldsymbol{z})$. Then there exist algebraic integers $\lambda_{il}$ $(1 \le i \le n,\ 0 \le l \le L)$, not all zero, and a rational function $R(\boldsymbol{z}) \in \overline{\mathbb{Q}}(\boldsymbol{z})$ such that

$$\sum_{i=1}^{n} \sum_{l=0}^{L} \lambda_{il} D_{j_i}^l G_i(\boldsymbol{z}) = R(\boldsymbol{z}). \tag{9}$$

Substituting 1 into $y_1, y_2, \dots, y_m$, we obtain

$$\sum_{i=1}^{n} \sum_{l=0}^{L} \lambda_{il} \ell_{ij_i}^l \sum_{h=1}^{\infty} h^l [h\chi] (\zeta_i M_i(\boldsymbol{x}))^h = R'(x_1, x_2, \dots, x_m) \in \overline{\mathbb{Q}}(\boldsymbol{x}).$$

We take a sufficiently large positive integer $t$ and attempt a specialization of the form

$$\boldsymbol{x} = (w^t, w^{t^2}, \dots, w^{t^m})$$

for a single variable $w$. Let $t_i = \sum_{j=1}^{m} \ell_{ij} t^j$ $(1 \le i \le n)$. Then

$$w^{t_i} = M_i(w^t, w^{t^2}, \dots, w^{t^m}).$$

We take $t$ so large that, if $M_i \ne M_j$, then $t_i \ne t_j$ $(1 \le i < j \le n)$ and that the denominator of $R^*(w) := R'(w^t, w^{t^2}, \dots, w^{t^m}) \in \overline{\mathbb{Q}}(w)$ does not vanish. Let $\lambda'_{il} = \lambda_{il} \ell_{ij_i}^l$ $(1 \le i \le n,\ 0 \le l \le L)$. Then we have

$$\sum_{i=1}^{n} \sum_{l=0}^{L} \lambda'_{il} \sum_{h=1}^{\infty} h^l [h\chi] (\zeta_i w^{t_i})^h = \sum_{k=0}^{\infty} a_k w^k = R^*(w),$$

where

$$\begin{aligned} a_k &= \sum_{\substack{1 \le i \le n \\ t_i \mid k}} \sum_{l=0}^{L} \lambda'_{il} \frac{k^l}{t_i^l} \left[ \frac{k\chi}{t_i} \right] \zeta_i^{k/t_i} \\ &= \sum_{\substack{1 \le i \le n \\ t_i \mid k}} \sum_{l=0}^{L} \lambda'_{il} \frac{k^l}{t_i^l} \left( \frac{k\chi}{t_i} - \left\{ \frac{k\chi}{t_i} \right\} \right) \zeta_i^{k/t_i} \end{aligned}$$

and $\{\cdot\}$ denotes the fractional part. Since $R^*(w) \in \overline{\mathbb{Q}}(w)$ and since $a_k$ $(k \geq 0)$ are algebraic integers, we can find

$$a_k = P_1(k)\xi_1^k + \cdots + P_M(k)\xi_M^k \qquad (k \geq k_0), \tag{10}$$

where $k_0$ is a sufficiently large integer, $P_1(x), \ldots, P_M(x) \in \overline{\mathbb{Q}}[x]$ and $\xi_1, \ldots, \xi_M$ are algebraic integers. Then by $a_k = O(k^{L+1})$, we see that $\xi_1, \ldots, \xi_M$ are roots of unity (cf. [10, proof of Theorem 3.4.8]). Let $N$ be a positive integer such that $\xi_1^N = \cdots = \xi_M^N = \zeta_1^N = \cdots = \zeta_n^N = 1$. Let $\{t'_1, \ldots, t'_r\}$ be the maximum subset of $\{t_1, \ldots, t_n\}$ with $t'_i \neq t'_j$ $(1 \leq i < j \leq r)$. Let $T_i = \{j \mid t_j = t'_i\}$ $(1 \leq i \leq r)$. Then $\zeta_j$ $(j \in T_i)$ are distinct for each $i$, since $\alpha_1, \ldots, \alpha_n$ are distinct. Put $s = t'_1 \cdots t'_r N$ and $s_i = s/t'_i$ $(1 \leq i \leq r)$. Noting that $\{1, \ldots, n\}$ is a disjoint union of $T_1, \ldots, T_r$, for any $k \geq k_0$ and for any fixed positive integer $h$, we see that

$$\begin{aligned}
a_{ks+h} &= \sum_{\substack{1\leq i\leq r \\ t'_i|h}} \sum_{l=0}^{L} \left( \sum_{j\in T_i} \lambda'_{jl}\zeta_j^{h/t'_i} \right) \frac{(ks+h)^l}{t_i'^l} \left( \frac{(ks+h)\chi}{t'_i} - \left\{ \frac{(ks+h)\chi}{t'_i} \right\} \right) \\
&= k^{L+1} \sum_{i=1}^{r} \lambda_{iL}^{(h)} s_i^{L+1}\chi \\
&\quad + k^L \sum_{i=1}^{r} \left( (L+1)h\lambda_{iL}^{(h)} s_i^L \chi/t'_i + \lambda_{i\,L-1}^{(h)} s_i^L \chi - \lambda_{iL}^{(h)} s_i^L \left\{ \frac{(ks+h)\chi}{t'_i} \right\} \right) \\
&\quad + \cdots - \sum_{i=1}^{r} \lambda_{i0}^{(h)} \left\{ \frac{(ks+h)\chi}{t'_i} \right\},
\end{aligned} \tag{11}$$

where

$$\lambda_{il}^{(h)} = \begin{cases} \sum_{j\in T_i} \lambda'_{jl}\zeta_j^{h/t'_i}, & \textit{if } t'_i|h, \\ 0, & \textit{otherwise}, \end{cases}$$

for $1 \leq i \leq r$. On the other hand, by (10) we have

$$a_{ks+h} = c_{L+1}^{(h)} k^{L+1} + c_L^{(h)} k^L + \cdots + c_0^{(h)}, \tag{12}$$

where $c_0^{(h)}, \ldots, c_{L+1}^{(h)}$ are algebraic numbers. We can take $L$ such that $\lambda'_{iL}$ $(1 \leq i \leq n)$ are not all zero.

Let $\boldsymbol{a} = (s_1\chi, s_2\chi, \ldots, s_r\chi)$. Renumbering $\alpha_1, \ldots, \alpha_n$, we may assume that $t'_1 < t'_2 < \cdots < t'_r$. Then $s_1 > \cdots > s_r$. Let $\boldsymbol{p} = \left(\lambda_{1L}^{(h)} s_1^L, \ldots, \lambda_{rL}^{(h)} s_r^L\right)$ and put $c_L^{(h)'} = \sum_{i=1}^r \left( (L+1)h\lambda_{iL}^{(h)} s_i^L \chi/t'_i + \lambda_{i\,L-1}^{(h)} s_i^L \chi \right) - c_L^{(h)}$.

**Lemma 3.** *If $\boldsymbol{p} \neq \boldsymbol{0}$, then there exists a real number $\tau_0$ such that*

$$\boldsymbol{p} \cdot (\boldsymbol{a}\tau_0 - ([\tau_0 s_1\chi], \ldots, [\tau_0 s_r\chi])) \neq c_L^{(h)'}.$$

***Proof.*** We see that

$$\begin{aligned}[0,1)^r &\ni \boldsymbol{a}\tau_0 - ([\tau_0 s_1\chi],\dots,[\tau_0 s_r\chi]) \\ &= \begin{cases} \boldsymbol{a}\tau_0, & \tau_0 \in [0, 1/(s_1\chi)), \\ \boldsymbol{a}\tau_0 - (*,\dots,*,\overset{i}{\check{1}},0,\dots,0), & \tau_0 \in [1/(s_i\chi), 1/(s_{i+1}\chi)), \end{cases}\end{aligned}$$

where $s_{r+1} = s_r/2$. Since $(*,\dots,*,\overset{i}{\check{1}},0,\dots,0)$ $(1 \le i \le r)$ are linearly independent and $\boldsymbol{p} \ne \boldsymbol{0}$, there exists an $i$ such that $\boldsymbol{p}\cdot(*,\dots,*,\overset{i}{\check{1}},0,\dots,0) \ne 0$. If $\boldsymbol{p}\cdot\boldsymbol{a} \ne 0$, then $\boldsymbol{p}\cdot\boldsymbol{a}\tau_0$ takes at least two values when $\tau_0$ varies in the interval $[0, 1/(s_1\chi))$. If $\boldsymbol{p}\cdot\boldsymbol{a} = 0$, then $\boldsymbol{p}\cdot(\boldsymbol{a}\tau_0 - ([\tau_0 s_1\chi],\dots,[\tau_0 s_r\chi]))$ takes at least two values when $\tau_0$ varies in the interval $[0, 1/(s_{r+1}\chi))$. Hence we can choose $\tau_0 \in \mathbb{R}$ such that

$$\boldsymbol{p}\cdot(\boldsymbol{a}\tau_0 - ([\tau_0 s_1\chi],\dots,[\tau_0 s_r\chi])) \ne c_L^{(h)\prime}. \qquad \square$$

**Lemma 4.** *For any real number $\tau$ there exists an increasing sequence $\{k_\nu\}_{\nu\ge 0}$ of positive integers such that*

$$\lim_{\nu\to\infty} (\{k_\nu s_1\chi\},\dots,\{k_\nu s_r\chi\}) = \boldsymbol{a}\tau - ([\tau s_1\chi],\dots,[\tau s_r\chi]),$$

*where each component of the left-hand side approaches the corresponding component of the right-hand side from the right.*

***Proof.*** First we consider the case of $\tau \ge 0$. For any $\varepsilon > 0$, there exist positive integers $p_\varepsilon$ and $q_\varepsilon$ such that

$$0 < q_\varepsilon\chi - p_\varepsilon < \frac{\varepsilon}{s_1\sqrt{r}},$$

since there are strictly increasing sequences $\{p_\nu\}_{\nu\ge 0}$ and $\{q_\nu\}_{\nu\ge 0}$ of positive integers such that $0 < q_\nu\chi - p_\nu < 1/q_\nu$. Let $\boldsymbol{e}_i = (0,\dots,0,\overset{i}{\check{-1}},0,\dots,0)$ $(1 \le i \le r)$. Then, every component of $q_\varepsilon\boldsymbol{a} + p_\varepsilon s_1\boldsymbol{e}_1 + \cdots + p_\varepsilon s_r\boldsymbol{e}_r$ is positive and less than $\varepsilon/\sqrt{r}$, and so $\|q_\varepsilon\boldsymbol{a} + p_\varepsilon s_1\boldsymbol{e}_1 + \cdots + p_\varepsilon s_r\boldsymbol{e}_r\| < \varepsilon$. Hence $\{\mu(q_\varepsilon\boldsymbol{a} + p_\varepsilon s_1\boldsymbol{e}_1 + \cdots + p_\varepsilon s_r\boldsymbol{e}_r) \mid \mu \in \mathbb{N}\}$ is distributed on the half-line $\boldsymbol{a}\mathbb{R}_{>0}$ with equal intervals of length less than $\varepsilon$. Therefore there exists a positive integer $\mu_\varepsilon$ such that

$$\|\mu_\varepsilon(q_\varepsilon\boldsymbol{a} + p_\varepsilon s_1\boldsymbol{e}_1 + \cdots + p_\varepsilon s_r\boldsymbol{e}_r) - \boldsymbol{a}\tau\| < \varepsilon \tag{13}$$

and every component of $\mu_\varepsilon(q_\varepsilon\boldsymbol{a} + p_\varepsilon s_1\boldsymbol{e}_1 + \cdots + p_\varepsilon s_r\boldsymbol{e}_r) - \boldsymbol{a}\tau$ is nonnegative. Let $\mu_\varepsilon q_\varepsilon = k_\varepsilon$ and $\mu_\varepsilon p_\varepsilon = k'_\varepsilon$. It is clear that $\boldsymbol{a}\tau - ([\tau s_1\chi],\dots,[\tau s_r\chi]) = \boldsymbol{a}\tau + [\tau s_1\chi]\boldsymbol{e}_1 + \cdots + [\tau s_r\chi]\boldsymbol{e}_r \in [0,1)^r$. By (13) we have

$$\begin{aligned}&\|k_\varepsilon\boldsymbol{a} + (k'_\varepsilon s_1 + [\tau s_1\chi])\boldsymbol{e}_1 + \cdots + (k'_\varepsilon s_r + [\tau s_r\chi])\boldsymbol{e}_r \\ &\quad -(\boldsymbol{a}\tau + [\tau s_1\chi]\boldsymbol{e}_1 + \cdots + [\tau s_r\chi]\boldsymbol{e}_r)\| < \varepsilon\end{aligned} \tag{14}$$

and hence we can choose $\varepsilon$ so small that $k_\varepsilon \boldsymbol{a}+(k'_\varepsilon s_1+[\tau s_1\chi])\boldsymbol{e}_1+\cdots+(k'_\varepsilon s_r+[\tau s_r\chi])\boldsymbol{e}_r \in (0,1)^r$. Since $k'_\varepsilon s_1+[\tau s_1\chi],\dots,k'_\varepsilon s_r+[\tau s_r\chi]\in\mathbb{Z}$, by the uniqueness of the fractional part, we see that

$$k_\varepsilon \boldsymbol{a}+(k'_\varepsilon s_1+[\tau s_1\chi])\boldsymbol{e}_1+\cdots+(k'_\varepsilon s_r+[\tau s_r\chi])\boldsymbol{e}_r = (\{k_\varepsilon s_1\chi\},\dots,\{k_\varepsilon s_r\chi\})\,.$$

Hence by (14) there exists an increasing sequence $\{k_\nu\}_{\nu\geq 0}$ of positive integers such that

$$\lim_{\nu\to\infty}(\{k_\nu s_1\chi\},\dots,\{k_\nu s_r\chi\}) = \boldsymbol{a}\tau-([\tau s_1\chi],\dots,[\tau s_r\chi])\,,$$

where each component of the left-hand side approaches the corresponding component of the right-hand side from the right.

Next we consider the case of $\tau<0$. For any $\varepsilon>0$, there exist positive integers $p_\varepsilon$ and $q_\varepsilon$ such that

$$-\frac{\varepsilon}{s_1\sqrt{r}}<q_\varepsilon\chi-p_\varepsilon<0,$$

since there are strictly increasing sequences $\{p_\nu\}_{\nu\geq 0}$ and $\{q_\nu\}_{\nu\geq 0}$ of positive integers such that $-1/q_\nu<q_\nu\chi-p_\nu<0$. Then $\{\mu(q_\varepsilon\boldsymbol{a}+p_\varepsilon s_1\boldsymbol{e}_1+\cdots+p_\varepsilon s_r\boldsymbol{e}_r)\mid \mu\in\mathbb{N}\}$ is distributed on the half-line $\boldsymbol{a}\mathbb{R}_{<0}$ with equal intervals of length less than $\varepsilon$. By the same way as above we can take an increasing sequence $\{k_\nu\}_{\nu\geq 0}$ of positive integers such that

$$\lim_{\nu\to\infty}(\{k_\nu s_1\chi\},\dots,\{k_\nu s_r\chi\}) = \boldsymbol{a}\tau-([\tau s_1\chi],\dots,[\tau s_r\chi])\,,$$

where each component of the left-hand side approaches the corresponding component of the right-hand side from the right. This completes the proof. $\square$

We assume that $\lambda_{iL}^{(h)}$ $(1\leq i\leq r)$ are not all zero. Then $\boldsymbol{p}\neq\boldsymbol{0}$. By Lemmas 3 and 4, we see that there exist a real number $\tau_0$ and an increasing sequence $\{k_\nu\}_{\nu\geq 0}$ of positive integers such that

$$\boldsymbol{p}\cdot(\boldsymbol{a}\tau_0-([\tau_0 s_1\chi],\dots,[\tau_0 s_r\chi]))\neq c_L^{(h)\prime} \tag{15}$$

and

$$\begin{aligned}\lim_{\nu\to\infty}(\{k_\nu s_1\chi\},\dots,\{k_\nu s_r\chi\}) &= \boldsymbol{a}\tau'_0-([\tau'_0 s_1\chi],\dots,[\tau'_0 s_r\chi]) \qquad (16)\\ &= (\{\tau'_0 s_1\chi\},\dots,\{\tau'_0 s_r\chi\}),\end{aligned}$$

where each component of the left-hand side approaches the corresponding component of the right-hand side from the right and $\tau'_0=\tau_0-h/s$. By (16) we see that

$$\begin{aligned}&\lim_{\nu\to\infty}(\{(k_\nu s+h)\chi/t'_1\},\dots,\{(k_\nu s+h)\chi/t'_r\})\\ =\ &\lim_{\nu\to\infty}(\{k_\nu s_1\chi+h\chi/t'_1\},\dots,\{k_\nu s_r\chi+h\chi/t'_r\})\\ =\ &(\{(\tau_0-h/s)s_1\chi+h\chi/t'_1\},\dots,\{(\tau_0-h/s)s_r\chi+h\chi/t'_r\})\\ =\ &(\{\tau_0 s_1\chi\},\dots,\{\tau_0 s_r\chi\}). \qquad (17)\end{aligned}$$

Since $\lim_{k\to\infty} a_{ks+h}/k^{L+1} = c_{L+1}^{(h)}$ by (12) and $\lim_{k\to\infty} a_{ks+h}/k^{L+1} = \sum_{i=1}^{r} \lambda_{iL}^{(h)} s_i^{L+1}\chi$ by (11), we have

$$c_{L+1}^{(h)} = \sum_{i=1}^{r} \lambda_{iL}^{(h)} s_i^{L+1}\chi. \tag{18}$$

By (12)

$$\lim_{k\to\infty} \frac{a_{ks+h} - c_{L+1}^{(h)} k^{L+1}}{k^L} = c_L^{(h)}.$$

On the other hand, by (11), (15), (17) and (18), we have

$$\begin{aligned}
&\lim_{\nu\to\infty} \frac{a_{k_\nu s+h} - c_{L+1}^{(h)} k_\nu^{L+1}}{k_\nu^L} \\
= &\lim_{\nu\to\infty} \frac{k_\nu^L \left(c_L^{(h)} + c_L^{(h)\prime} - \sum_{i=1}^{r} \lambda_{iL}^{(h)} s_i^L \{(k_\nu s + h)\chi/t_i'\}\right) + \cdots}{k_\nu^L} \\
= &\lim_{\nu\to\infty} \left(c_L^{(h)} + c_L^{(h)\prime} - \boldsymbol{p}\cdot(\{(k_\nu s+h)\chi/t_1'\},\ldots,\{(k_\nu s+h)\chi/t_r'\})\right) \\
\neq\ & c_L^{(h)},
\end{aligned}$$

which is a contradiction. Hence we see that $\lambda_{iL}^{(h)} = 0$ $(1 \le i \le r)$ for any positive integer $h$. Hence for $h = t_i'k$ with $k \ge 0$ and for $i$ with $1 \le i \le r$ we have

$$\sum_{j\in T_i} \lambda_{jL}' \zeta_j^k = 0.$$

Since $\zeta_j$ $(j \in T_i)$ are distinct, by non-vanishing of the Vandermonde determinant, we see that $\lambda_{iL}' = 0$ $(1 \le i \le n)$, which is a contradiction, and the proof of the theorem is completed. □

## References


[1] Y. Z. Flicker. *Algebraic independence by a method of Mahler*, J. Austral. Math. Soc. Ser. A **27** (1979), 173–188.

[2] E. Hecke. *Über analytische Funktionen und die Verteilung von Zahlen mod Eines*, Abh. Math. Sem. Hamburg **1** (1921), 54–76.

[3] J. H. Loxton and A. J. van der Poorten. *Arithmetic properties of certain functions in several variables III*, Bull. Austral. Math. Soc. **16** (1977), 15–47.

[4] J. H. Loxton and A. J. van der Poorten. *Algebraic independence properties of the Fredholm series*, J. Austral. Math. Soc. Ser. A **26** (1978), 31–45.

[5] K. Mahler. *Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen*, Math. Ann. **101** (1929), 342–366.

[6] D. W. Masser. *Algebraic independence properties of the Hecke-Mahler series*, Q. J. Math. **50** (1999), 207–230.

[7] K. Nishioka. *Evertse theorem in algebraic independence*, Arch. Math. **53** (1989), 159–170.

[8] K. Nishioka. *Note on a paper by Mahler*, Tsukuba J. Math. **17** (1993), 455–459.

[9] K. Nishioka. *Algebraic independence of Mahler functions and their values*, Tohoku Math. J. **48** (1996), 51–70.

[10] K. Nishioka. *Mahler functions and transcendence*, LNM **1631**, Springer (1996).

[11] T. Tanaka and Y. Tanuma. *Algebraic independence of the values of the Hecke-Mahler series and its derivatives at algebraic numbers*, Int. J. Number Theory (accepted).